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THE PROPAGATION OF ELECTROMAGNETIC WAVES IN A STATISTICALLY INHOMOGENEOUS MEDIUM --
I: A Critique of the Current Theory

by

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THE PROPAGATION OF ELECTROMAGNETIC WAVES IN A STATISTICALLY INHOMOGENEOUS MEDIUM*

I: A Critique of the Current Theory

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Abstract

The theory of propagation of electromagnetic waves in a statistically inhomogeneous medium is examined for the models of the medium: (1) the discrete scatterer model, (2) the perturbed continuum model. The theory is based on the assumption that the lifetime of a configuration of the system is long compared with the period of the primary time harmonic wave. It is generally assumed also that averages of the field obtained by the ensemble method are equivalent to long time averages over the time series that the actual field constitutes. These assumptions are examined and their implications are pointed out. It is shown that there is a complete correspondence between the methodology used for the discrete scatterer model and the perturbed continuum model.

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I. Introduction.

The subject of propagation of electromagnetic waves in a statistically inhomogeneous medium is of interest over the entire range of the spectrum from the low frequency radio region to the X-ray region. A statistically inhomogeneous medium, as the name implies, is one whose structure and properties fluctuate in a random manner about some value or state that is uniform, at least locally. The special interest in such a medium as far as electromagnetic wave propagation is concerned centers on the scattering phenomena associated with the fluctuations in the structure of the medium. The motivations of the interest are quite varied. The communications engineer is interested in ionospheric and tropospheric scattering of radio waves from the standpoint of the medium as a communication channel. The geophysicist is interested in radio wave and optical scattering by the various portions of the atmosphere for what it tells him about the structure and dynamics of the atmosphere. The physicist and chemist utilize the scattering phenomena as a tool to study the structure of matter. While the range of interests cover an extremely broad spectrum, the basic electromagnetic problem is the same in each instance to the extent that only classical processes, and no quantum mechanical processes, are involved.

The purpose of this paper is to explore this common ground and to examine some of the basic assumptions and methodology. It is hardly possible to review all of the work nor do justice to the extensiveness of the subject in this paper. The literature is very extensive for each of the scientific and engineering disciplines and the spectral region of their special concern. However, the differences between the problems in the different regions of the spectrum are more with respect to the character of the fluctuations in the structure of the medium than the scattering process per se. The reader interested in ionospheric and tropospheric scattering of radio waves will find a review article by A. D. Wheelon(1) and the special issue of the Proceedings of the Institute of Radio Engineers on "Scatter Propagation"(2) very useful. Two Russian books by Chernov(3) and Tatarski(4) give admirable expositions of the many techniques that have been developed in the field. A ready introduction to optical and X-ray scattering as a tool in the study of the structure of fluids is available in a book by H. S. Green(5), while a comprehensive paper by M. Lax(6) provides a general exposition of scattering theory in the physicist's language.

II. Classes of Problems.

The types of problems that are usually considered may be classified according to the model that is used for the medium. We may classify them accordingly as (1) problems of a perturbed continuum, and (2) problems of propagation through an assemblage of discrete scatterers. The perturbed continuum model assumes a priori that the medium can be characterized by constitutive parameters ε, μ, σ that are point functions of space and time. Their temporal variation is associated with the fluctuations in the structure of the medium. It is assumed implicity that the statistical regime is such that the constitutive parameters have mean values that are independent of the averaging interval and that over intervals that are long compared with the period of the incident waves the structure of the medium is sensibly constant. In other words, the time variation of the medium is a slow function compared with the oscillations in the wave.

The discrete scatterer model considers the medium to be an assemblage of scatterers. The total field is expressed as a sum of the contributions made by the individual scatterers. When only the primary wave is considered as the driving function exciting a given scatterer, we are dealing with the single scattering treatment. When the contributions of the other members of the assemblage, that is, their scattered waves, to the driving function exciting a given member are considered, we are dealing with a multiple scattering treatment. Because of the complexity of the latter in its most general form, multiple scattering analyses are usually carried out on a rather limited basis. The statistical fluctuations of the medium are expressed in terms of the statistical character of the configuration of scatterers and the scattering phenomena are averaged over ensembles of configurations.

An essential ingredient of the treatment is the assumption that each configuration of the scatterers, or of the perturbed continuum, is stationary over intervals that are long compared with the period of the primary wave. Doppler shifts, even on a classical basis, are, accordingly, being neglected entirely. While the statistical regime leads to incoherent scattering by virtue of the randomness of phase relationships between different parts of the medium there is no broadening or dispersion in frequency associated with the fluctuations.

It should be stated further that we are dealing only with the classical regime. Such effects as Compton scattering, resonance absorption and reradiation, the Raman effect and other quantum mechanical processes are ignored completely in this discussion.

III. Discussion of the Discrete Scatterer Problem.

The purpose of this section is to review the treatment of the wave propagation through an assemblage of discrete scatterers. This is the model used in the study of optical and X-ray scattering, in the discussion of artificial dielectrics, and, in general, in the development of the macroscopic field equations for material media from a microscopic picture of the medium. In the case of artificial dielectrics, where only fixed configurations such as arrays are usually involved, there is, of course, no problem of fluctuation scattering.

The first comprehensive treatment of wave propagation through an assembly of randomly distributed scatterers seems to have been carried out by L. Foldy[7] for the case of a scalar wave field and isotropic scatterers. M. Lax[6] presumably extended the treatment to anisotropic elements but the effectiveness of his treatment of anisotropy is open to question. We shall consider for the present Foldy's model. It will serve to identify the essential considerations and problems.

With reference to Figure 1, we consider an assembly of N scattering elements distributed over a volume V. The incident wave, that is, the wave which would exist in the absence of scattering elements, is considered to be a scalar wave having a space dependence $\psi_o(\vec{r})$ and time dependence $e^{j\omega t}$. The vector $\vec{r}$ is the position vector of a field point relative to the origin O. The wave function satisfies the scalar Helmholtz equation

$$\nabla^2\psi_o + k_o^2\psi_o = 0 \tag{1}$$

where k_o is the propagation constant in free space. On the assumption that the configuration of scattering elements is fixed spatially, the total field $\psi(\vec{r})$ which arises by superposition of scattered waves on the primary field is also a time harmonic field and in the region between scattering elements actually satisfies the same equation (1) as the primary wave. Since the elements are isotropic point scatterers, their scattered fields are given by

$$\psi_\ell^{(s)} = A_\ell \frac{e^{-jk_o|\vec{r}-\vec{r}_\ell|}}{|\vec{r}-\vec{r}_\ell|} \tag{2}$$

where $\vec{r}_\ell$ is the position of the ℓ^{th} scatterer. The total field given by

$$\psi(\vec{r}) = \psi_o(\vec{r}) + \sum_\ell A_\ell \frac{e^{-jk_o|\vec{r}-\vec{r}_\ell|}}{|\vec{r}-\vec{r}_\ell|} \tag{3}$$

thus varies rapidly with position within the volume V, having singularities at each scattering point $\vec{r}_\ell$. The "smoothed" field $\langle\psi(\vec{r})\rangle_s$ is, however, a continuously varying field corresponding to some equivalent continuous medium in which a wave would travel with a modified propagation constant $\bar{k}$. The smoothed field is obtained by averaging wave fields such as are given by Eq. (2) for an assemblage over a statistical ensemble of assemblages. The result of such averaging, as Foldy shows, has both a coherent component, that is, a component of the total field that bears a definite phase relation to the primary field, and an incoherent compenent.

In the study of fluctuation scattering we are more concerned with the far zone field of the assemblage. Following the usual procedures for the far zone approximation, we obtain the field of the ℓ^{th} scatterer to be

$$A_\ell \frac{e^{-jk_oR}}{R} e^{j\vec{k}\cdot\vec{r}_\ell}$$

where $R = |\vec{r}|$, the distance of the far field point from the origin O, and $\vec{k}$ is the propagation vector directed from the origin to the field point. The total field is

$$\psi = \psi_o + \frac{e^{-jk_oR}}{R} \sum_\ell A_\ell e^{j\vec{k}\cdot\vec{r}_\ell} \tag{4}$$

We can now direct our attention to some of the particular features of the problem. The amplitude of excitation of the ℓ^{th} scatterer is the sum of the response A_ℓ^o to the primary wave ψ_o and the response $\bar{A}_\ell$ to the scattered waves coming from all the other scatterers. Let the response of a scattering element to a wave of unit amplitude, with a phase reference of zero at the scattering element be g_ℓ. Because the elements are isotropic point scatterers the response to a spherical wave is the same as that to a plane wave since a spherical wave is sensibly plane in the neighborhood of the scattering element. Thus,

$$A_\ell^o = g_\ell \left|\psi_o\right| e^{-j\vec{k}_o \cdot \vec{r}_\ell}$$

and

$$\bar{A}_\ell = g_\ell \sum_{m \neq \ell} A_m \frac{e^{-jk_o\left|\vec{r}_m - \vec{r}_\ell\right|}}{\left|\vec{r}_m - \vec{r}_\ell\right|} = g \sum_{m \neq \ell} A_m G(\vec{r}_m, \vec{r}_\ell)$$

The far field is then

$$\psi = \psi_o + \frac{e^{-jk_oR}}{R} \sum_\ell g \left|\psi_o\right| e^{j(\vec{k}-\vec{k}_o) \cdot \vec{r}_\ell}$$
$$+ \frac{e^{-jk_oR}}{R} \sum_\ell \sum_{m \neq \ell} g A_m G(\vec{r}_m, \vec{r}_\ell)\, e^{j\vec{k} \cdot \vec{r}_\ell} \tag{5}$$

It is interesting to note that by resolving A_m into two parts A_m^o and $\bar{A}_m$ and substituting into Eq. (5), and subsequently iterating the procedure we obtain an expansion of the scattered field in successive orders of multiple scattering interactions. The second term on the right-hand side of Eq. (5) represents the single scattering expression in which only the contribution of the primary field to the excitation of the scatterers is considered. The successive scattering orders involve the interparticle distances in corresponding inverse degree:

$$\frac{e^{-jk_o\left|\vec{r}_m - \vec{r}_\ell\right|}}{\left|\vec{r}_m - \vec{r}_\ell\right|}; \quad \frac{e^{-jk_o\left\{\left|\vec{r}_m - \vec{r}_\ell\right| + \left|\vec{r}_n - \vec{r}_m\right|\right\}}}{\left|\vec{r}_m - \vec{r}_\ell\right| \cdot \left|\vec{r}_n - \vec{r}_m\right|}; \text{ etc.}$$

When the averaging process is carried out over ensembles the successive orders decrease at least as rapidly as the inverse power of the mean interparticle distance and, in fact, more so because of the random phase factors. This accounts for the success of the single scattering approximation for even rather dense distributions.

However, before discussing any averaging process we should consider several basic points. One has already been made, namely, that the response of a given scatterer is the same to the scattered waves (per unit amplitude) from the other scatterers as to the primary wave. This is true because of the isotropic point scatterer model that we have taken; the response characteristic is, further, independent of the particular configurations and poses no problem to subsequent averaging processes. The second point is that representations for the scattered fields are based on fixed scatterers and, therefore, the response of each element to both the primary wave and the scattered waves of the other elements is time harmonic with a time dependence $e^{j\omega t}$. It is, therefore, essential that the statistical regime is such that any given configuration has a "lifetime" that is long compared with the period of the primary wave. The "lifetime" is a fundamental consideration in two respects, one, in so far as the motion of the particles is neglected in the single scattering formulation in the manner of writing the response of a scatterer to the primary wave, and the other pertaining to the calculation of multiple scattering in which retardation effects associated with the relative motions of the particles are neglected. It is not directly obvious that the second involves the same order of approximation as the first, for the second consideration depends in a different way on the breadth of the Fournier spectrum arising from the motion of the particles.

As was stated earlier, the randomness of the medium and the fluctuation scattering is brought into the treatment by regarding each assemblage or configuration as a member of a statistical ensemble. The ensemble is characteristically expressed in terms of a distribution function.

$$P\left(\vec{r}_1, \vec{r}_2, ----, \vec{r}_N\right)$$

expressing the probability of finding the particles at positions $\vec{r}_1$, $\vec{r}_2$, ----, $\vec{r}_N$. The probability function is usually assumed to be independent of time, that is, the fluctuation in the configuration is regarded to be a stationary process. The stationary process is itself not a sufficient criterion for neglecting the retardation effects in the scattered waves; it is necessary to consider the time scale of the fluctuations as was noted previously.

Next, let us consider more general types of scatterers. Some of the most general work on this subject has been done by V. Twersky. The reader is referred to his summary paper(8) in the U. R. S. I. monograph from the 1960 General Assembly of U. R. S. I. for details. Twersky's approach is to set up the solution to the scalar wave equation by Green's function methods for an N-body system. The time dependence is again $e^{j\omega t}$ and he assumes that all responses are corresponding functions of time; the time dependence is thus split away from the problem. The Green's function technique expresses the total field as the sum of the primary wave and scattered waves that are given in terms of integrals over the boundaries of the scattering bodies. In each integral there appear both the primary field over the body and the contributions to the field from all other bodies, considering, of course, a fixed configuration of the assembly. Twersky then proceeds to average his general representation over an ensemble of configurations to discuss the statistical behavior of the system and the fluctuation scattering.

It is more convenient for our present purposes to express the procedure in somewhat different language that is suited particularly to discussion of the far zone field of the system. Referring to Figure 2, where we consider a single scattering body and an incident plane wave, we note that we can choose any reference point in the body as a local origin O_L and with respect to that origin the far zone field takes the form of the field of a directive point source, namely,

$$\Psi_s^o = \frac{e^{-jk_o\bar{R}}}{\bar{R}} \quad F^o(\vec{k}_o, \vec{k}) \tag{6}$$

Here $\vec{k}_o$ is the propagation vector of the incident wave and $\vec{k}$ is the propagation vector in the direction of observation from the origin. The function $F^o(\vec{k}_o, \vec{k})$ gives the angular dependence of the far zone field, that is, it expresses the directivity. It contains the interference effects among contributions from the elements of volume of the scattering body occasioned by the finite extent of the body and, therefore, phase differences between wavelets from different elements of volume. It is important to remember that the space factor, as $F^o(\vec{k}_o, \vec{k})$ is generally named, depends on the aspect presented by the scattering body to the incident plane wave. It is, in general, a complex quantity with a phase angle depending on phase reference for the incident wave and on the structure of the scatterer. The basic definition for F^o will be taken to be relative to a plane wave whose phase reference is at the local origin in the body.

Suppose, now we have an assemblage of scattering bodies, for simplicity assumed to be all alike. Equivalent points in each of the bodies are assumed to be taken as local origins. Then, in the single scattering approximation we can obtain the total scattered field in the far zone of the configuration by the same superposition procedure as before with the space factors playing the role of the amplitudes A^o of the isotropic point scatterer case. Thus, the total field in the far zone is given by

$$\Psi = \Psi_o + \frac{e^{-jkR}}{R} \sum_{\ell} F_{\ell}^{o}(\vec{k}_o, \vec{k})\, e^{-j(\vec{k}_o - \vec{k}) \cdot \vec{r}_{\ell}} \tag{7}$$

where R is the distance from the general origin of reference for the system to the point of observation and $\vec{r}_{\ell}$ is the position vector from the general origin to the local reference origin in the ℓ^{th} scattering body.

Although, formally, Eq. (7) is the identical with Eq. (5) in the single scattering approximation, that is, with the last term in Eq. (5) being neglected, we are confronted here with a new complication. In the case of isotropic point scatterers all the space factors, given there by $g|\Psi_o|$ are the same and there is no orientation factor involved. In the present case, however, the factors $F_{\ell}^{o}(\vec{k}_o, \vec{k})$ do depend on orientation and when we consider averages over ensembles we must keep in mind that the F_{ℓ}^{o}'s differ from one sample of the ensemble to another. The statistical representation of the system will have to be given by a more general probability function

$$P(\vec{r}_1, s_1, \vec{r}_2, s_2, \text{------} \vec{r}_N, s_N)$$

in which $\vec{r}_\ell$ is the position of the ℓ^{th} scatterer and s_ℓ is a variable or set of variables describing the orientation of the body. The fluctuation scattering arises from both the fluctuation in the relative positions of the bodies and the fluctuation in their orientations.

The response of a given scattering body to a more general type of field, for example, such as the field due to all the other scatterers is a much more intricate matter. The near zone structure of the field produced by any one scattering body, which we may have to consider for dense distributions, is neither isotropic nor dependent on $|\vec{r}-\vec{r}_\ell|$ in the simple form of Eq. (2). It is not possible to represent the interaction between scatterers by functions of only the distance between them. However, since any arbitrary field can be resolved into a spectrum of plane waves the far zone scattered field of a body excited by an arbitrary field is a superposition of components such as given by Eq. (6) from all the component plane waves, and, therefore, the result takes the form

$$\psi_s = \frac{e^{-jk_o \bar{R}}}{\bar{R}} \mathcal{F}(\vec{k}). \tag{8}$$

The notation $\mathcal{F}(\vec{k})$ signifies a function of the direction of the field point from the local origin. The form of $\mathcal{F}(\vec{k})$ when we consider the interaction between the scattering bodies will depend on the particular pair of bodies, their relative orientations and the distance between them. Thus, the total contribution of the ℓ^{th} scatterer to the far zone field due to excitation by the scattered waves from the other N-1 scatterers is

$$\psi_s = \frac{e^{-jk_o \bar{R}}}{\bar{R}} \sum_{m\neq\ell}^{N-1} \mathcal{F}_{m\ell}(\vec{k}; \vec{r}_m-\vec{r}_\ell; s_m, s_\ell)$$

where we have written $\mathcal{F}_{m\ell}(\vec{k}; \vec{r}_m-\vec{r}_\ell; s_m, s_\ell)$ to show that the response is a function of the relative positions of the bodies and their orientations. The total field, in the far zone, is now given by

$$\Psi = \Psi_o + \frac{e^{-jk_oR}}{R}\left\{\sum_{\ell} F_{\ell}^{\,o}(\vec{k}_o, \vec{k})\, e^{-j(\vec{k}_o-\vec{k})\cdot\vec{r}_{\ell}} + \sum_{\ell} e^{j\vec{k}\cdot\vec{r}_{\ell}} \sum_{m\neq\ell} \mathcal{F}_{m\ell}(\vec{k};\, \vec{r}_m-\vec{r}_{\ell};\, s_m,\, s_{\ell})\right\} \tag{9}$$

where R is now the distance from the general origin O to the field point.

Twersky's treatment of the problem to which we referred earlier constructs the interaction terms as well as the single scattering response amplitudes contained in the quantities $F_{\ell}^{\,o}$ and $\mathcal{F}_{m\ell}$ in the form of integrals over the boundaries of the scatterers. The formulation which we have used here is simply a rearrangement of his and, of course, does not give explicitly the form of the response amplitudes. It should be noted that, in principle, each $\mathcal{F}_{m\ell}$ can be resolved into two parts, (1) the excitation of the ℓ^{th} scatterer by the first order wave from the m^{th} scatterer which is excited by the incidence of the primary plane wave on the m^{th} scatterer, and (2) the excitation of the ℓ^{th} scatterer by the higher ordered waves from the m^{th} scatterer excited by the incidence of scattered waves from all of the other bodies on the m^{th} scatterer. The multiple scattering term of Eq. (9) can then be developed into successive orders of multiple scattering in a manner similar to that described for the system of isotropic point scatterers. There is unfortunately no simple way of writing the general expression of the field of a single scatterer excited by a plane wave. When, however, the distance between scattering bodies is sufficiently large it is possible to resort to far zone representations of the form of Eq. (6) to express the scattered waves incident on one body from another.

All of the remarks we made about the averaging of the field over an ensemble of configurations for the case of isotropic point scatterers apply to the more general system. Each configuration of the ensemble must have a lifetime that is long compared with the period of the incident wave. The lifetime of a configuration is a separate problem that, in fact, must be considered before making any generalizations about the averaging over ensembles and the biggest gap in the theory of propagation through a random distribution of scatterers is the lack of analyses of the actual statistical regimes under which physical systems operate.

In the foregoing we limited our discussion to scalar wave fields. The extension to the electromagnetic field requires only formal changes in what we have written for the case of the general scattering bodies. The treatment of isotropic point scatterers is excluded by the electromagnetic field. The elementary scatterer for the latter is a dipole and the simple spherical waves of the previous discussion must be replaced by dipole fields with the scalar amplitude A_1 approximately replaced by a vector dipole moment. In the case of the general scattering body the far zone scattered wave amplitude $F_\ell^{\,o}(\vec{k}_o, \vec{k})$ becomes a complex vector quantity that depends not only on the aspect of the body with respect to the incident wave front but also with respect to the polarization of the incident wave. If the incident wave is

$$\vec{E}_i = \vec{E}_o \, e^{j(\omega t - \vec{k}_o \cdot \vec{r})}$$

the formal representation of the field in the far zone of the scattering configuration is

$$\vec{E} = \vec{E}_o \, e^{-j\vec{k}_o \cdot \vec{R}} + \frac{e^{-jk_oR}}{R} \left\{ \sum_\ell \vec{F}_\ell^{\,o}(\vec{k}_o, \vec{k}) \, e^{-j(\vec{k}_o - \vec{k}) \cdot \vec{r}_\ell} + \sum_\ell e^{j\vec{k} \cdot \vec{r}_\ell} \sum_{m \neq \ell} \vec{\mathcal{F}}(\vec{k}; \vec{r}_m - \vec{r}_\ell, s_m s_\ell) \right\} \tag{10}$$

the factor $e^{j\omega t}$ having been factored out.

The polarization of the scattered wave depends markedly on the structure and shape of the scattering body unless the latter is a sphere of isotropic material. The treatment of multiple scattering becomes hopelessly complicated and useful results can be obtained only from the single scattering component of the expansion. The commonly used theory of the scattering of light and X-rays by gases is based on the single scattering approximation. In general, even in single scattering, there are depolarization effects and the fluctuation scattering has both amplitude and polarization fluctuations. The latter are lost in the background, however, in experimental work because the incident radiation is usually both incoherent and randomly polarized.

We should note another level of approximation that is frequently introduced into the single scattering treatment. It is a form of a general procedure, the Born approximation, used in many instances to develop an approximate solution to a scattering problem. The process of scattering by a body of permittivity ε can be regarded as radiation from dipoles induced in the body under the influence of the primary wave. The polarization vector $\vec{P}$ in the body is given by

$$\vec{P} = (\varepsilon - \varepsilon_0)\vec{E} \tag{11}$$

where $\vec{E}$ is the total field within the body. An element of volume dV of the body is a dipole of moment $\vec{P}dV$ and the scattered field is the sum of the component fields set up by the elementary dipoles. The form of the Born approximation to which we have just alluded is to take the field within the body to be the incident field $\vec{E}_i$ to a first approximation. That is,

$$\vec{P} \approx (\varepsilon - \varepsilon_0)\vec{E}_i \tag{11a}$$

Under this approximation, when the incident field is linearly polarized, all the elementary dipoles making up the scatterer are oriented in the same direction and, consequently, their fields are all polarized in the same direction. The space factor $\vec{F}^o(\vec{k}_o, \vec{k})$ is then, within this approximation, a linearly polarized vector oriented with respect to $\vec{E}_i$ at each point in the far field in the same way as the field of dipole is oriented with respect to the dipole moment.

In the case of X-ray scattering, for example, where the scatterer is an atom or a molecule, the induced polarization arises from a displacement of the electrons from their reference equilibrium positions under the action of the incident field, that is, within the Born approximation. The calculation of the induced polarization proceeds exactly along the same lines as the calculation of the induced polarization in the ionosphere in the Appleton-Hartree treatment of propagation through the ionosphere. Further reference to this treatment will be made later when we discuss the theory of scattering by a perturbed continuum.

IV. Ensemble Averages of the Scattered Field

The field calculated by Eq. (10) is for a fixed configuration and its time dependence is obtained simply by multiplying in the factor $e^{j\omega t}$ and taking the real part. The observed field, on the other hand, is not simply time harmonic but undergoes random fluctuations associated with the fluctuations in the configuration of scatterers. As was stated earlier the point of view from which one generally proceeds is that statistical behavior of the configuration is formulated in terms of an ensemble of configurations with which is associated a probability function giving the probability of a given configuration. Correspondingly, the field given by Eq. (10) is to be regarded as a sample of an ensemble of fields, the probability of a given field being the probability of the configuration from which it arises. The basic assumption is that the averages of the actual field, the square of the absolute value of the field, etc. over sufficiently long intervals of time are equal to the corresponding averages of the sample fields over the ensemble.

The two most important averages are those of the field, $\langle E\rangle$, and of the square of the modulus $\langle |E|^2\rangle$:

$$\langle\vec{E}\rangle = \int\!\cdots\!\int \vec{E}\, P(\vec{r}_1, s_1, \ldots, \vec{r}_N, s_N)\, d\vec{r}_1 \ldots d\vec{r}_N\, ds_1 \ldots ds_N \tag{12}$$

$$\langle |E|^2\rangle = \int\!\cdots\!\int |E|^2\, P(\vec{r}_1, s_1, \ldots, \vec{r}_N, s_N)\, d\vec{r}_1 \ldots d\vec{r}_N\, ds_1 \ldots ds_N \tag{13}$$

Writing the field given by Eq. (10) as

$$\vec{E} = \vec{E}_i + \vec{E}_s$$

where $\vec{E}_s$ represents the scattered wave,

$$\vec{E}_s = \frac{e^{-jk_oR}}{R} \sum_{\ell} \vec{F}_\ell^{\,o}(\vec{k}_o, \vec{k})\, e^{-j(\vec{k}_o-\vec{k})\cdot\vec{r}_\ell} + \frac{e^{-jk_oR}}{R} \sum_{\ell}\sum_{m\neq\ell} \vec{\mathcal{F}}(\vec{k}; \vec{r}_m-\vec{r}_\ell; s_m; s_\ell)\, e^{j\vec{k}\cdot\vec{r}_\ell}$$

we have

$$\langle \vec{E} \rangle = \vec{E}_i + \langle \vec{E}_s \rangle \tag{14}$$

$$\langle |E|^2 \rangle = \langle \vec{E} \cdot \vec{E}^* \rangle = |E_i|^2 + \vec{E}_i^* \cdot \langle \vec{E}_s \rangle + \vec{E}_i \cdot \langle E_s^* \rangle + \langle |E_s|^2 \rangle \tag{15}$$

Eq. (15) can be rewritten in the form

$$\langle |E|^2 \rangle = |\vec{E}_i + \langle \vec{E}_s \rangle|^2 + \langle |E_s|^2 \rangle - \langle |\vec{E}_s|^2 \rangle \tag{16}$$

The form of Eq. (16) has been regarded by Lax[6] as the basis for interpreting $\langle \vec{E}_s \rangle$ as the coherent component of the scattered field. The first part of the right-hand side of (16) implies an interaction or interference between $\vec{E}_i$ and $\langle \vec{E}_s \rangle$, that is, that $\langle \vec{E}_s \rangle$ has a determinate phase relationship with $\vec{E}_i$ while the second part

$$\langle |E_s| \rangle^2 - |\langle E_s \rangle|^2 \tag{17}$$

is the standard form of the mean square fluctuation of a quantity associated with the statistical ensemble. It appears, however, to the writer that this interpretation of $\langle \vec{E}_s \rangle$ requires further validation vis à vis the time series $\vec{E}(\vec{r}, t)$ that the actual field constitutes and which in the time domain cannot be resolved readily into a time harmonically varying component with fixed phase on which is superimposed a fluctuating component. The time average of a segment of the time series over no matter how long a time interval is not a unique quantity. The time average of $|\vec{E}(\vec{r}, t)|$ over a sufficiently long interval is a unique statistical quantity but that must be related separately to the ensemble average $|\langle E_s \rangle|$.

The situation is quite different with respect to $\langle |E|^2 \rangle$. Except for a multiplicative constant $|E|^2$ is the average intensity in the field, the average being over a number of cycles of the harmonic variation. The time factor $e^{j\omega t}$ disappears from the situation and need not be considered further in interpreting $\langle |E|^2 \rangle$. Thus the ensemble average $\langle |E|^2 \rangle$ and the average of the time series $|E(\vec{r}, t)|^2$ should be equivalent for time averages over sufficiently long intervals. The only limitation on the relationship resides in the primary assumption that the lifetime of a configuration is long compared with the period of the primary wave.

The explicit evaluation of (12) and (13) depends, of course, on the details of the distribution function and the structure of the scatterers. Various examples are to be found in the work of Foldy[7] and in the theory of optical and X-ray scattering (see Green[5]). There are a few points to which we wish to call attention here in relation to the evaluation of $\langle |E_i|^2 \rangle$ in the far zone. It is observed that Eq. (15) for the average contains terms $|\vec{E}_i|^2$ and $\vec{E}_i \cdot \langle \vec{E}_s^{\,*} \rangle$ and $\vec{E}_i^{\,*} \cdot \langle \vec{E}_s \rangle$. In most scattering experiments the incident field is in a collimated beam produced by a directive antenna or lens system and the far zone field is observed at angles outside the region where $\vec{E}_i$ is significant. Thus, except for small angle scattering, that is, forward scattering and small angles about the principal direction of the incident wave, $\vec{E}_i \approx 0$ and we are concerned solely with $\vec{E}_s$. The latter has been expressed in Eq. (10) in two parts,

$$\vec{E}_s = \vec{E}_s^{\,o} + \vec{E}_s(m,\ell) \tag{18}$$

where $\vec{E}_s^{\,o}$ is the single scattering component given in terms of the quantities $\vec{F}_\ell^{\,o}(\vec{k}_o, \vec{k})$ and $\vec{E}_s(m,\ell)$ is the multiple scattering component. Thus,

$$\langle |\vec{E}_s|^2 \rangle = \langle |E_s^{\,o}|^2 \rangle + \langle \vec{E}_s^{\,o} \cdot \vec{E}_s^{\,*}(m,\ell) + \vec{E}_s^{\,o*} \cdot E_s(m,\ell) \rangle \\ + \langle |\vec{E}_s(m,\ell)|^2 \rangle \tag{19}$$

It is very difficult to say much of anything about the last two terms. It must be remembered that the averaging process involves averaging over the orientation variables s_ℓ, s_m as well as the position variables. If there are no forces coupling the orientations of scatterers the contributions made by the terms involving multiple scattering may be expected to be small compared with the single scattering terms even when there are forces that effect correlations between the positions of the scatterers, for we are averaging over random orientations of the component vectors. Of course, the importance of the multiple scattering diminishes also with decreasing average number density of scattering bodies.

In so far as the single scattering term is concerned we observe that

$$R^2 \langle |\vec{E}_s^{\,o}|^2 \rangle = \int \cdots \int \sum_{\ell} \sum_{m} \vec{F}_{\ell}^{\,o} \cdot \vec{F}_m^{\,o*} \; e^{-j(\vec{k}_o - \vec{k}) \cdot (\vec{r}_\ell - \vec{r}_m)} \quad (20)$$
$$\times \, P(\vec{r}_1, s_1, \ldots, \vec{r}_N, s_N) d\vec{r}_1 ds_1 \ldots d\vec{r}_N ds_N$$

The quantities $\vec{F}_{\ell}^{\,o}$ and $\vec{F}_{\ell}^{\,o} \cdot \vec{F}_m^{\,o*}$ do not depend on the positions of the scatterers but do depend on their orientations. If the distribution of orientations is independent of that over positions, the distribution function factors into

$$P(\vec{r}_1, s_1, \ldots\ldots, \vec{r}_N, s_N) = P_r(\vec{r}_1, \ldots, \vec{r}_N) \; P_s(s_1, \ldots, s_N)$$

In that case the integration over the orientation variables can be carried out separately and what is of importance as far as orientations are concerned is

$$\langle \vec{F}^{\,o} \cdot \vec{F}_m^{\,o*} \rangle_s = \int \cdots \int \vec{F}_{\ell}^{\,o} \cdot \vec{F}_m^{\,o*} P_s \, ds_1 \ldots ds_N$$

This yields

$$R^2 \langle |E_s^o|^2 \rangle = \sum_{\ell} \sum_{m} \langle \vec{F}_{\ell}^{\,o} \cdot \vec{F}_m^{\,o*} \rangle_s \int \cdots \int e^{-j(\vec{k}_o - \vec{k}) \cdot (\vec{r}_\ell - \vec{r}_m)}$$
$$\times \, P_r(\vec{r}_1, \ldots, \vec{r}_N) \, d\vec{r}_1 \ldots d\vec{r}_N$$

The sum can be resolved into two parts, one for $\ell = m$ and the other $\ell \neq m$. The result is then

$$R^2\langle |E_s^o|^2\rangle = N\langle|\vec{F}^o|^2\rangle_s$$
$$+\sum_{\ell}\sum_{m\neq\ell}\langle\vec{F}_\ell^o\cdot\vec{F}_m^{o*}\rangle_s \iint e^{-j(\vec{k}_o-\vec{k})\cdot(\vec{r}_\ell-\vec{r}_m)} p(\vec{r}_\ell,\vec{r}_m)\,d\vec{r}_\ell\,d\vec{r}_m \tag{21}$$

where

$$p(\vec{r}_\ell,\vec{r}_m) = \int\cdots\int{}'' P_r(\vec{r}_1,\ \ldots\ldots,\ \vec{r}_N)\,d\vec{r}_1\ \cdots\ d\vec{r}_N$$

the double prime signifying that integration is over all variables other than $\vec{r}_\ell$ and $\vec{r}_m$. In cases such as a fluid in statistical equilibrium the two particle correlation probability is a function of only the distance $|\vec{r}_\ell-\vec{r}_m|$.

If there is no correlation of the orientations of the scatterers, we have

$$\langle\vec{F}_\ell^o\cdot\vec{F}_m^{o*}\rangle_s = \langle\vec{F}_\ell^o\rangle_s\cdot\langle\vec{F}_m^{o}\rangle_s = |\langle\vec{F}_\ell^o\rangle_s|^2 = |\langle\vec{F}^o\rangle_s|^2$$

and

$$R^2\langle|E_s^o|^2\rangle = N\langle|\vec{F}^o|^2\rangle_s + |\langle\vec{F}^o\rangle|^2\sum_{\ell}\sum_{m\neq\ell}\iint e^{-j(\vec{k}_o-\vec{k})\cdot(\vec{r}_\ell-\vec{r}_m)} p(\vec{r}_\ell,\vec{r}_m)\,d\vec{r}_\ell\,d\vec{r}_m \tag{22}$$

The double sum can actually be simplified further when p, $(\vec{r}_\ell,\vec{r}_m)$ is a function $p(|\vec{r}_\ell-\vec{r}_m|)$ but that is not of special interest for this discussion. It is interesting to note that the first term in Eq. (22) is proportional to the number N of the scatterers. This is expressive of the incoherent component of the scattering for when the scattered fields are completely independent of one another owing to complete random phases, the total intensity is the sum of the intensities of the component fields. The second term, on the other hand, involves interference effects between component fields.

V. The Perturbed Continuum Model

In this model the medium is considered to be a continuum whose density, and, consequently its electromagnetic properties, undergoes random variations. The fluctuations may be due to turbulence as in the case of the troposphere or they may be the statistical thermal fluctuations in a fluid in thermal equilibrium. In any case the starting point of the analysis is taken to be the macroscopic equations such as Maxwell's equations for a medium with macroscopic and continuous, or step-wise continuous, constitutive parameters or the various wave equations for continuous media. It is important to remember that all of such equations are already equations for averages of fields which in the microscopic picture of the medium are rapidly varying functions of position and in the microscopic domain are also randomly varying according to the statistical behavior of the assemblage of atomic, electronic, and molecular components that make up the medium. The development of the field equations for a continuum from the microscopic level was first carried out by H. Lorentz[9]; a more modern phrasing of the problem and the averaging process will be found in the excellent monograph by Rosenfeld[10].

The electromagnetic properties of the medium are characherized by the constitutive parameters, ε, μ, σ which as we have stated are regarded, aside from fluctuations, to be continuous or step-wise continuous functions of position. As we know, these parameters, strictly speaking, are defined as functions of frequency, that is, for fields varying harmonically with time. The fluctuations in the properties of the medium are assumed to be representable by constitutive parameters having statistical properties. Only the electric permittivity is considered usually. It is assumed that

$$\varepsilon(\vec{r}, t) = \overline{\varepsilon}(\vec{r}) + \Delta\varepsilon(\vec{r}, t) \qquad (23)$$

where $\overline{\varepsilon}(r)$ is the mean value, generally assumed to be the permittivity of the steady state of the continuum about which the structure fluctuates, while $\Delta\varepsilon(\vec{r}, t)$ is a random function of position and time.

It is well to recognize the implicit assumptions on which the model is based. It implies that the statistical regime is such that the average

$$\overline{\varepsilon}(\vec{r}) = \frac{1}{\tau} \int_{t_o}^{t_o + \tau} \varepsilon(\vec{r}, t)dt \qquad (24)$$

is independent to t_o and τ for "sufficiently" long intervals τ. The relation between $\overline{\varepsilon}$ and the permittivity of the unperturbed steady state continuum has been investigated by K. Budden[11] for the ionosphere on the basis of the Appleton-Hartree type of analyses of wave propagation in the ionosphere.

He showed that $\hat{\varepsilon}$ differs from the permittivity of the unperturbed continuum and that the effect is significant for the treatment of long wave propagation.

It is also assumed that fluctuations are slow on the time scale of the microscopic picture of the medium. This requires some discussion. The continuum and the appropriate field equations for time varying phenomena in the continuum such as wave propagation are derived from the microscopic picture along lines very similar to our discussion of propagation through an assemblage of discrete scatters. The continuum is arrived at by taken time averages over the microscopic fields and distributions of discrete elements. The time varying field and microscopic field equations represent the variations in time averages over intervals in time that are long compared with intervals over which the microscopic representatives are averaged, that is, long compared with the mean lifetime of the configurations of discrete elements. Thus, fluctuations in the continuum constitute an overlay on the fluctuations in the microscopic fields and microscopic structure of the medium. When the fluctuations involved are associated with thermal effects the fluctuations of the continuum are, of course, themselves expressions of fluctuations in the microscopic structure. There yet remains the problem of relating the fluctuations in the continuum and the treatment of the situation in terms of already averaged microscopic parameters to the averaging process carried out in passing from the microscopic domain to the continuum.

The third consideration is the relation between the statistical properties of $\varepsilon(\vec{r}, t)$ as a function of time, for say a given position $\vec{r}$, with the statistical properties of $\varepsilon(\vec{r}, t)$ as a function of $\vec{r}$ for a given t. This involves both the question of multiple scattering and the structural features of the medium such as long range forces and short range forces between atoms, electrons, ions, etc. Budden's work, to which we referred previously, constitutes a partial treatment of this matter. There is need for a far more comprehensive study of this aspect of the subject.

Finally we must take note that when the explicit problem of propagation through the perturbed continuum is treated the primary field is taken to have a harmonic time variation $e^{j\omega t}$ and the total field is likewise taken to have a time dependence $e^{j\omega t}$ on which the fluctuations are superimposed. As was done in the treatment of propagation through discrete scatterers the harmonic time dependence is factored out and the fluctuating medium is treated in terms of an ensemble of perturbed configurations, each of which has a lifetime that is long compared with the period of the primary wave. Again, the scattered field appears at the same frequency as the primary field; Doppler effects and the associated dispersion in frequency of the scattered field are neglected. One recognizes here perhaps more clearly the problem to which we referred earlier, namely, that of establishing relationships between the "lifetime" of the perturbed continuum configurations and "lifetimes" of microscopic configurations on which the continuum itself is based.

VI. Calculation of the Scattered Field.

The purpose of this section is to examine the rather standard procedure for calculating the fluctuation scattering and to show the relation to the methods employed in the treatment of an assemblage of discrete scatterers. Consider, as shown by Figure 3, a volume V of the medium within which the permittivity is assumed to be given by Eq. (1). The incident wave is again

$$\vec{E}_i = \vec{E}_o e^{j(\omega t - \vec{k}_o \cdot \vec{r})}$$

It is assumed that the relation between the polarization and the field intensity that holds for a medium in the steady state, that is, the unperturbed medium, namely

$$\vec{P} = (\varepsilon - \varepsilon_o)\vec{E} \tag{25}$$

can be extended to the fluctuating medium. Consequently, the fluctuation gives rise to a fluctuating polarization vector,

$$\vec{P} + \Delta\vec{P} = (\vec{\varepsilon} + \Delta\varepsilon - \varepsilon_o)\vec{E}, \tag{26}$$

or

$$\Delta\vec{P} = \Delta\varepsilon(\vec{r}, t)\vec{E}, \tag{27}$$

and the element of volume dV at the point $\vec{r}$ is a radiating dipole $\Delta\vec{P}\,dV$. The "fluctuation scattering" is ascribed to $\Delta\vec{P}$ and is assumed to be separable from the coherent scattering (which represents just the ordinary propagation of the wave) on this basis.

The field vector $\vec{E}$ appearing in Eq. (25) and Eq. (27) is strictly speaking the total field at the given point. It consists of an average component in relation to the statistical variations) and a fluctuating component associated, first of all, with the matter in the element of volume V at the given point and, secondly, with scattered waves coming from other elements of volume of the region. It is usual to assume at this point, as was done, for example, by Booker and Gordon[12] in their classical paper on scatter propagation, that the field $\vec{E}$ can be replaced to a first approximation by the incident field $\vec{E}_i$. This neglects all orders of multiple scattering and employs the Born approximation which we have already described in Section III. The Born approximation is a reasonable one when the body is a "soft" scatterer, that is, when

$$\frac{\varepsilon-\varepsilon_o}{\varepsilon_o} \ll 1$$

Correspondingly, the Born approximation for the perturbed continuum is based in the assumption that the fluctuations are such that

$$\left| \frac{\Delta \varepsilon}{\varepsilon} \right| \ll 1 \tag{28}$$

On the basis of the Born approximation the fluctuating component of the dipole moment associated with the element of volume V is

$$\Delta \vec{P} dV = \Delta \varepsilon \ \vec{E}_o \ e^{j(\omega t - \vec{k}_o \cdot \vec{r})} dV \tag{29}$$

We can proceed by two different directions: In one, we do precisely the same thing as in the treatment of the assembly of discrete scatterers, namely, consider the statistical problem in terms of an ensemble of perturned configurations. On the assumption that each sample of the ensemble has a lifetime that is long compared with the period of the incident wave, the elementary dipoles given by Eq. (29) can be regarded as having a time dependence $e^{j\omega t}$ and the radiation can be computed on that basis. In the other direction, we consider the elementary dipole moment to be a general function of time as determined by the explicit form that $\Delta \varepsilon (\vec{r}, t)$ takes in combination with the time dependence of the incident wave. The field is then set up in terms of retarded polarization vectors as prescribed by the general form of the solution to the general time dependent field equations. The scattered field is thus obtained as a time series, and averages are obtained subsequently in the time domain rather than over an ensemble of perturbed configurations.

VII. The Method of Ensemble Averaging.

Let us first proceed along the line of ensemble averages. The far field of harmonically varying dipole is give by

$$\vec{E} = \left(\frac{-k^2}{4\pi\varepsilon}\right) p \quad \sin\theta \; \frac{e^{-jk\bar{R}}}{\bar{R}} \; \vec{a}_\theta \tag{30}$$

where $\vec{a}_\theta$ is a unit vector shown in Fig. 4. Thus the field due to the element dV is

$$d\vec{E}_s = \left(\frac{-k^2}{4\pi\varepsilon}\right) \Delta\varepsilon E_o \; e^{-j\vec{k}_o \cdot \vec{r}} \; dV \; \sin\theta \; \frac{e^{-jk_o\bar{R}}}{\bar{R}} \; \vec{a}_\theta \tag{31}$$

where $\vec{a}_\theta$ again is shown in Fig. 4b, and $\bar{R}$ is the distance from the element of volume to the field point. By the usual procedure for developing the far zone field, the factor $e^{-jk\bar{R}}/\bar{R}$ can again be transformed to

$$\frac{e^{-jk_o\bar{R}}}{\bar{R}} \simeq \frac{e^{-jk_oR}}{R} \cdot e^{j\vec{k} \cdot \vec{r}} \tag{32}$$

where R is the distance from the origin O to the field point and $\vec{k}$ is the propagation vector in the scattered wave in the direction from O to the field point. We have then,

$$d\vec{E}_s = \left\{ \left(\frac{-k^2}{4\pi\varepsilon}\right) E_o \sin\theta \, dV \; e^{-j(\vec{k}_o - \vec{k}) \cdot \vec{r}} \right\} \vec{a}_\theta \, \frac{e^{-jk_oR}}{R}$$

The total scattered field is then

$$\vec{E}_s = \frac{e^{-jk_oR}}{R} \sum \left\{ \left(\frac{-k^2}{4\pi\varepsilon}\right) \Delta\varepsilon \, E_o \sin\theta \, dV \, \vec{a}_\theta \right\} e^{-j(\vec{k}_o - \vec{k}) \cdot \vec{r}}$$

$$= \frac{e^{-jk_oR}}{R} \left\{ \int \left(\frac{-k^2}{4\pi\varepsilon}\right) \Delta\varepsilon \, E_o \sin\theta \; e^{-j(\vec{k}_o - \vec{k}) \cdot \vec{r}} \; dV \right\} \vec{a}_\theta \tag{34}$$

It is recognized at once that Eq. (34) is the equivalent of the single scattering component $\vec{E}_s^{\,o}$ of Eq. (10) that was developed for the discrete scatterer model. The quantity

$$\frac{-k^2}{4\pi\varepsilon} \Delta\varepsilon \, E_o \sin\theta \, dV \, \vec{a}_\theta$$

is the equivalent of $\vec{F}^o(\vec{k}_o, \vec{k})$, the space factor of a single scatterer; the summation over the discrete scatterers has passed into an integral for the continuous distribution.

As was done in the discussion of the discrete scatterer model the scattered power is computed for a fixed perturbed configuration:

$$R^2 |E_s|^2 = \frac{k^4 |E_o|^2}{16\pi^2 \varepsilon^2} \sin^2\theta \iint \Delta\varepsilon(\vec{r}) \, \Delta\varepsilon(\vec{r}')^* \, e^{-j(\vec{k}_o - \vec{k}) \cdot (\vec{r} - \vec{r}')} \, d\vec{r} \, d\vec{r}' \qquad (35)$$

Introducing the vector $\vec{\xi} = \vec{r} - \vec{r}'$ we transform the above to

$$R^2 |E_s|^2 = \frac{k^4}{16\pi^2 \varepsilon^2} |E_o|^2 \sin^2\theta \int \left\{ \int \Delta\varepsilon(\vec{r}') \, \Delta\varepsilon(\vec{r}' + \vec{\xi}) \, d\vec{r}' \right\} e^{-j(\vec{k}_o - \vec{k}) \cdot \vec{\xi}} \, d\vec{\xi} \qquad (36)$$

The inner integral is except for a constant the autocorrelation function of the permittivity perturbation used by Booker and Gordon, namely,

$$p(\vec{\xi}) = \frac{1}{\overline{(\Delta\varepsilon)^2} \, V} \int_V \Delta\varepsilon(\vec{r}') \, \Delta\varepsilon(\vec{r}' + \vec{\xi}) \, d\vec{r}' \qquad (37)$$

where $\overline{(\Delta\varepsilon)^2}$ is mean square value of the perturbation over the scattering volume. Hence,

$$R^2 |E_s|^2 = \frac{V k^4 |E_o|^2 \sin^2\theta}{16\pi^2} \overline{\left(\frac{\Delta\varepsilon}{\varepsilon}\right)^2} \int p(\vec{\xi}) \, e^{-j(\vec{k}_o - \vec{k}) \cdot \vec{\xi}} \, d\vec{\xi} \qquad (38)$$

In order to obtain $R^2 \langle |E_s| \rangle^2$ it is necessary to average this expression over the ensemble of perturbed configurations. In the present language the configuration is characterized by the autocorrelation function p $(\vec{\xi})$ and the ensemble average over Eq. (38) is obtained by merely using the ensemble average of p $(\vec{\xi})$. Hence,

$$R^2|E_s|^2 = \frac{Vk^4|E_o|^2 \sin^2\theta}{16\pi^2} \overline{\left(\frac{\Delta\varepsilon}{\varepsilon}\right)^2} \int_V \langle p(\vec{\xi})\rangle \, e^{-j(\vec{k}_o - \vec{k})\cdot\vec{\xi}} \, d\vec{\xi} \qquad (39)$$

It is important to keep in mind that this result is based on the single scattering approximation and the Born approximation. If the Born approximation had been used in Section IV to compute the space factors $\vec{F}_\ell^{\,o}$, Eq. (20) would have involved the autocorrelation function of $(\varepsilon - \varepsilon_o)$, which in the discrete scatterer case is a stepwise continuous function of position. The correspondence between the treatment of the discrete scatterer model and that of the perturbed continuum model is actually complete.

Before leaving the ensemble averaging discussion it is of interest to note the significance of the integral in Eq. (39). Letting

$$\vec{K} = \vec{k}_o - \vec{k}$$

we obtain a function of $\vec{K}$,

$$\mathcal{R}(\vec{K}) = \int_V \overline{p(\vec{\xi})} \, e^{-j\vec{K}\cdot\vec{\xi}} \, d\vec{\xi} \qquad (40)$$

namely the Fourier transform of the function $\overline{p(\vec{\xi})}$.† The scattered power is thus within a factor of $\sin^2\theta$ the segment of the Fourier transform of $\overline{p(\vec{\xi})}$ corresponding to the physically admissible region of $\vec{K}$ given by $\vec{k}_o - \vec{k}$.

VIII. The Time Series Method.

Let us consider now the second procedure referred to in Section VI, namely, that of obtaining explicity the time dependence of the scattered field in terms of the assumed time variation of the medium. The starting point should in this case be the general time dependent field equations. The program for dealing with Maxwell's equations for time varying media, that

† The function p $(\vec{\xi})$ is regarded to be zero when $\vec{\xi}$ lies outside the scattering volume.

is, the construction of solutions in terms of potentials and the construction of the differential equations satisfied by the potentials, has, to this writer's knowledge, never been carried out. There are many difficulties and, in fact, when the changes in the structure of the medium are very large or very rapid it is necessary even to examine the basic definitions of the constitutive parameters. However, when the variations are small and are slow compared with the frequency of the primary wave it is reasonable to retain the concept of permittivity and to use Eq. (23) as a representation of a time varying medium.

The regime of fluctuations of small amplitude was treated in a very interesting paper by H. Staras[13]. Except for the method of setting up the basic expressions for the scattered field the following is based on Staras' work. Our starting point is again the representation of the perturbed medium by a distribution of dipoles discussed in Section VI, and we use again the Born approximation so that the dipole moment of an element of volume is

$$\Delta\vec{P}dV = \Delta\varepsilon(\vec{r}, t)\,\vec{E}_i = \Delta\varepsilon(\vec{r}, t)\,\vec{E}_o\, e^{j[\omega t - \vec{k}_o \cdot \vec{r}]}$$

where as before $\vec{k}_o$ is the propagation vector of the primary plane wave, and $\vec{r}$ is the position vector of the element dV. Now, however, in contrast with the ensemble method we retain the time dependence of $\Delta\varepsilon(\vec{r}, t)$ and, consequently, the dipole moment of the element of volume has a time dependence that is a composite of the fluctuations associated with $\Delta\varepsilon$ and the harmonic variation of the incident field. To get the field we use the general solution for time varying sources, namely, the retarded potentials. Since the source is given as a dipole it is convenient to use the Hertz vector,

$$\vec{\Pi} = \frac{1}{4\pi\varepsilon_o} \int_V \frac{[\Delta P]}{\bar{R}}\, dV \qquad (41)$$

where $[\Delta\vec{P}]$ is the retarded value of $\Delta\vec{P}\,(t)$, that is, evaluated at $t - \bar{R}/C$, where $\bar{R}$ is the distance from dV to the field point and $\underline{c}$ is the velocity of propagation in free space. Inserting the expression for $\Delta\vec{P}$ we obtain

$$\vec{\Pi} = \frac{1}{4\pi\varepsilon_o} \int_V \frac{\Delta\varepsilon\left(\vec{r},\ t - \frac{\bar{R}}{c}\right)\ \vec{E}_i\left(\vec{r},\ t - \frac{\bar{R}}{c}\right)}{\bar{R}}\, dV \tag{42}$$

or,

$$= \frac{1}{4\pi\varepsilon_o} \int_V \frac{\Delta\varepsilon\left(\vec{r},\ t - \frac{\bar{R}}{c}\right)\ \vec{E}_o\ e^{j\left[\omega\left(t - \frac{R}{c}\right) - \vec{k}_o \cdot \vec{r}\right]}}{\bar{R}}\, dV \tag{43}$$

The electric and magnetic fields are obtained from the Hertz vector by the relation

$$\left.\begin{aligned} \vec{E}_s &= \vec{\nabla}(\vec{\nabla}\cdot\vec{\Pi}) - \mu\varepsilon_o \frac{\partial^2\vec{\Pi}}{\partial t^2} \\ \vec{H}_s &= \varepsilon_o \vec{\nabla} \times \frac{\partial\vec{\Pi}}{\partial t} \end{aligned}\right\} \tag{44}$$

the space derivatives being with respect to the coordinates of the field point. Eq. (42) was developed by Staras by a more formal procedure starting from a generalized wave equation for the Hertz vector for a time varying medium whose permittivity has the form of Eq. (23). Staras showed that Eq. (42) is the first order approximation to the Hertz vector developed in a power series in terms of a smallness parameter measuring the amplitude of the fluctuations.

It is observed from the form of Eq. (44) that the complete expressions for $\vec{E}_s$ and $\vec{H}_s$ involve time derivatives of the function $\Delta\varepsilon(\vec{r},\ t)$. However, for the regime under consideration the variation of $\Delta\varepsilon$ is very slow compared with the variation associated with the time factor $e^{j\omega t}$ of the primary wave. If we neglect the time derivatives of $\Delta\varepsilon(\vec{r},\ t)$, as Staras did in his work, the far zone field can be written down at once from the elementary dipole field given previously in Eq. (30). That is, the dipole sources are regarded as being essentially time harmonic dipoles with slowly varying amplitudes determined by $\Delta\varepsilon(\vec{r},\ t)$. To this order of approximation, then, the far zone scattered field is given by

$$\vec{E}_s = \frac{-k^2 \sin\theta}{4\pi\epsilon_o} E_o e^{j\omega t} \left\{ \int_V \Delta\varepsilon\left(\vec{r}, t - \frac{\bar{R}}{c}\right) e^{-j\vec{k}_o \cdot \vec{r}} \frac{e^{-jk_o \bar{R}}}{\bar{R}} dV \right\} \vec{a}_\theta \quad (45)$$

with θ and $\vec{a}_\theta$ having the same definitions as before.

Again, with the condition that $\Delta\varepsilon$ is a slowly varying function of time, the power flow in the far field can be obtained by the procedure for a purely harmonically varying field, that is,

$$P = \frac{1}{2} \operatorname{Re} (\vec{E} \times \vec{H}^*) \cdot \vec{a}_R = \text{const. } |E_s|^2$$

where $\vec{a}_R$ is a unit vector from the origin to the field point. Furthermore, we can make the far zone approximation in so far as the factor $e^{-jk_o\bar{R}}/\bar{R}$ in the integrand is concerned. The only requirement for the latter is that the distance from the origin to the field point is large compared with every linear dimension of the scattering volume V. We obtain then

$$R^2 |E_s|^2 = \text{const. } |E_o|^2 \sin^2\theta \int_V \int_V \Delta\varepsilon\left(\vec{r}, t - \frac{\bar{R}}{c}\right) \Delta\varepsilon^*\left(\vec{r}', t - \frac{\bar{R}'}{c}\right) + e^{-j(\vec{k}_o - \vec{k}) \cdot (\vec{r} - \vec{r}')} dV\, dV' \quad (46)$$

where R is the distance from the origin to the field point and $\vec{k}$ is the propagation vector (associated with a pure time harmonic field) in the direction from the origin to the field point.

In this way the power flow, per unit solid angle now, is obtained as a time dependent function corresponding to the fluctuations of $\Delta\varepsilon$. The foregoing development shows clearly the approximations involved in generating this time series representation of the power function. The average is now to be obtained as a time average over an interval that is long compared with the fluctuations; more precisely

$$R^2 \langle |E_s| \rangle^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R^2 |E_s(t)|^2 \, dt$$

$$\doteq \text{const.} \; |E_o|^2 \sin^2\theta \int_V \int_V \left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Delta\varepsilon\left(\vec{r}, \, t - \frac{\bar{R}}{c}\right) \Delta\varepsilon^*\left(\vec{r}\,', \, t - \frac{\bar{R}'}{c}\right) dt \right]$$

$$\times \; e^{-j(\vec{k}_o - \vec{k}) \cdot (\vec{r} - \vec{r}\,')} \, dV \, dV' \qquad (47)$$

The quantity

$$N(\vec{r}, \vec{r}\,') = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Delta\varepsilon\left(\vec{r}, \, t - \frac{\bar{R}}{c}\right) \Delta\varepsilon^*\left(\vec{r}\,', \, t - \frac{\bar{R}'}{c}\right) dt \qquad (48)$$

can be rewritten as

$$N(\vec{r}, \vec{r}\,') = \lim_{T\to\infty} \frac{1}{2T} \int_{-T-\frac{\bar{R}}{c}}^{T-\frac{\bar{R}}{c}} \Delta\varepsilon(\vec{r}, \tau) \, \Delta\varepsilon^*\left(\vec{r}\,', \tau + \frac{\bar{R}-\bar{R}'}{c}\right) d\tau$$

$$\doteq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Delta\varepsilon(\vec{r}, \tau) \, \Delta\varepsilon^*\left(\vec{r}\,', \tau + \frac{\bar{R}-\bar{R}'}{c}\right) d\tau$$

If $\Delta\varepsilon$ does not change measurably over the time interval $\frac{\bar{R}-\bar{R}'}{c}$, the latter can be neglected in the integral and we have the basic time correlation function

$$N(\vec{r}, \vec{r}\,') = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Delta\varepsilon(\vec{r}, \tau) \Delta\varepsilon^*(\vec{r}\,', \tau) \, d\tau \qquad (49)$$

The average power is then given by

$$R^2 \langle |E_s|^2 \rangle = \text{const.} \; |E_o|^2 \sin^2\theta \iint_V N(\vec{r}, \vec{r}\,') \, e^{-j(\vec{k}_o - \vec{k}) \cdot (\vec{r} - \vec{r}\,')} \, dV dV' \qquad (50)$$

In this way the statistical property of the scattered field is expressed directly in terms of a statistical property of the medium. The time correlation function now enters into determining the average power rather than the average value of the spatial autocorrelation function of the permittivity.

When the statistical structure of $\Delta \varepsilon(\vec{r}, t)$ is such that

$$N(\vec{r}, \vec{r}\,') = N(\vec{r} - \vec{r}\,'),$$

that is, the time correlation is a function of only the relative positions the integral of Eq. (50) can be treated in a manner similar to that whereby Eq. (36) was transformed into Eq. (39). In that case,

$$R^2 \langle |E_s|^2 \rangle = \text{const.} \; |E_o|^2 \sin^2\theta \int_V N(\vec{\xi}) \, e^{-j(\vec{k}_o - \vec{k}) \cdot \vec{\xi}} \, d\xi \qquad (51)$$

and the average power is essentially proportional to the Fourier transform of the time correlation.

The time series method has some more satisfying features than the ensemble method. The line of development shows more clearly the conditions that must prevail in making the approximations involved. The requirement on the "lifetimes" of perturbed configurations shows itself more explicitly in the requirement that the function $\Delta\varepsilon(\vec{r}, t)$ must remain sensibly constant over a number of oscillations of the primary wave in the computation of $|E_s|^2$.

IX. Summary

The theory of propagation of electromagnetic waves in randomly varying media has been examined for the discrete scatterer model and the perturbed continuum model. The equivalence of the methodology employed in both instances was shown together with the underlying assumptions. A primary assumption underlying the treatment of both models is that the statistical regime is such that a configuration of the system has a lifetime that is long compared with the period of the primary waves. The problem of great importance that remains to be solved is that of rapidly moving scatterers or rapidly varying media. The solution of the problem will be especially significant in studying highly turbulent atmospheres and plasmas in which the kinetic temperatures are very large.

It has been shown also that the current theory is limited to a regime of fluctuations of small amplitude. Multiple reflection effects have yet to be evaluated properly even for this regime and the long lifetime situation. In the case of discrete scatterers the effect of orientation and of the distribution of orientations must be investigated farther. In the case of the perturbed continuum model a more careful analysis must be made of the mean value of the permittivity and the effective field within the medium from which the polarization should be determined rather than using the Born approximation.

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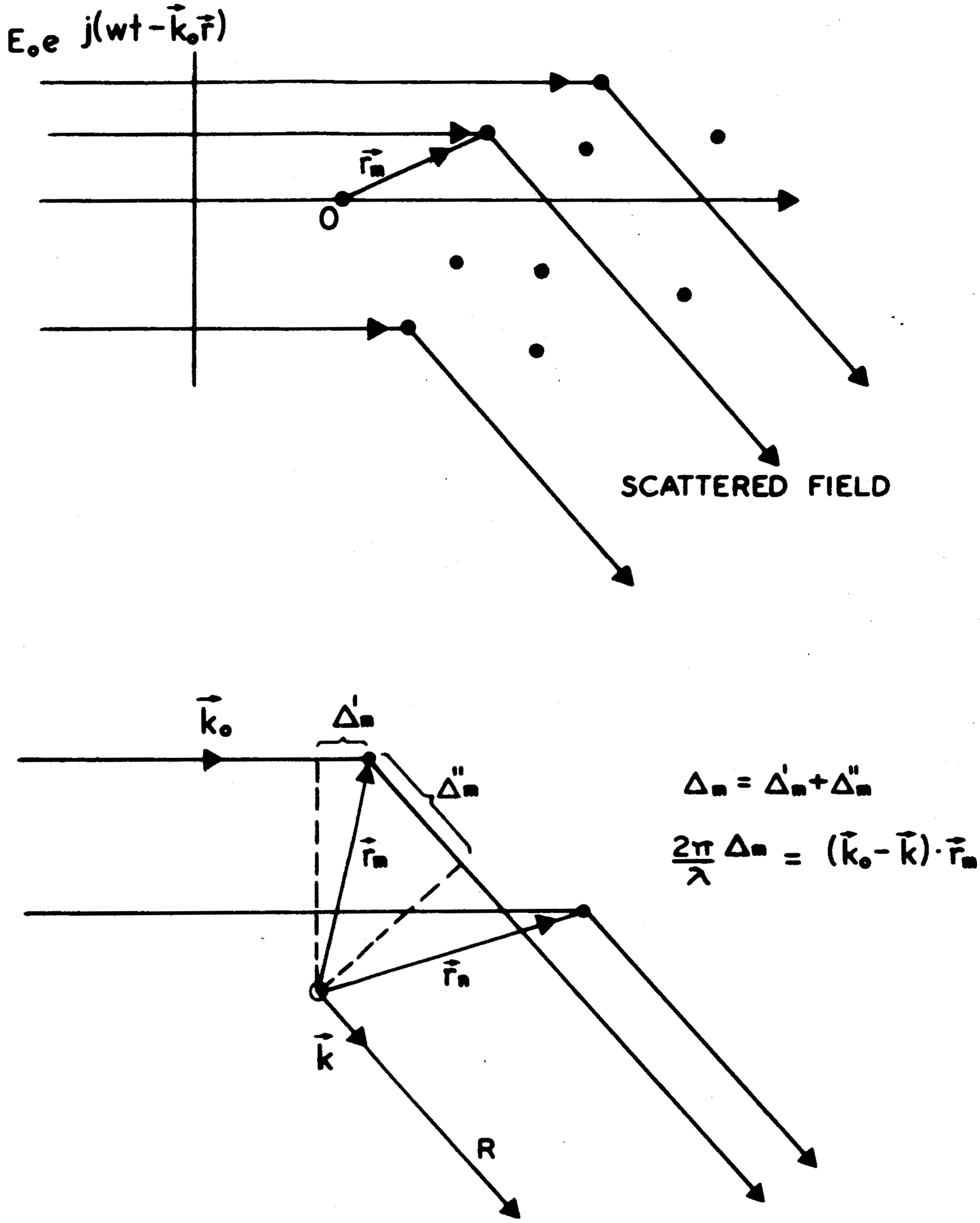

FIG. 1. SCATTERING BY A CONFIGURATION OF POINT SCATTERERS.

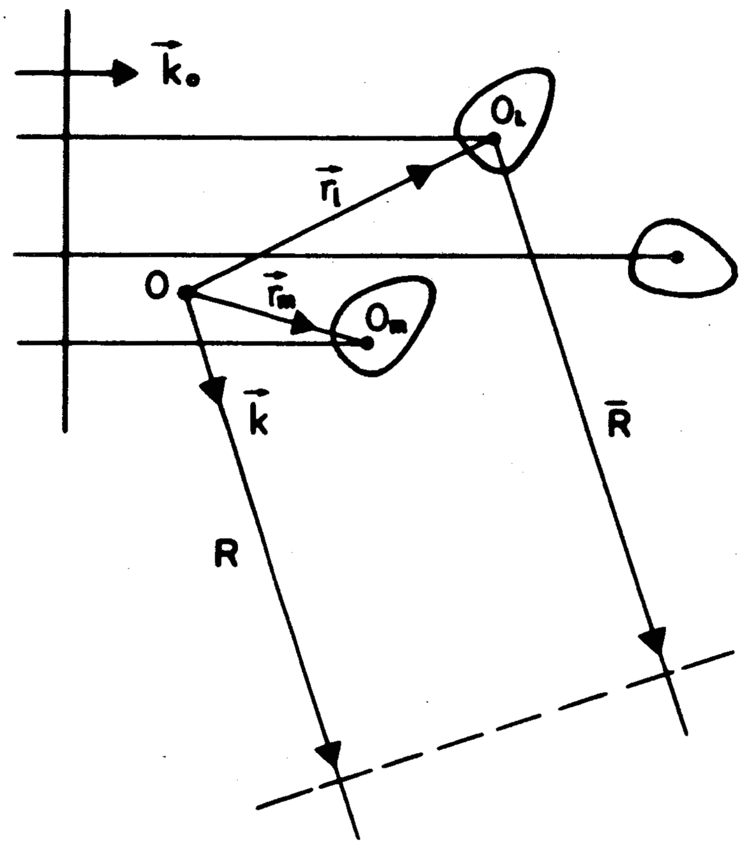

FIG. 2. SCATTERING BY A CONFIGURATION OF SCATTERING ELEMENTS OF NON-NEGLIGIBLE SIZE.

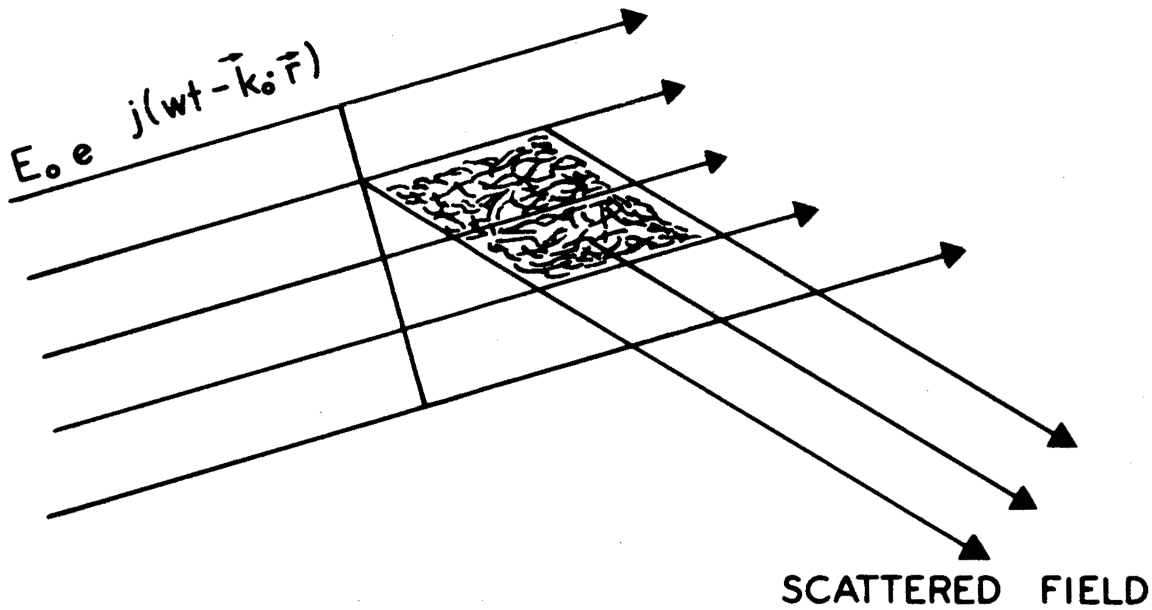

FIG. 3. SCATTERING BY A FLUCTUATING MEDIUM.

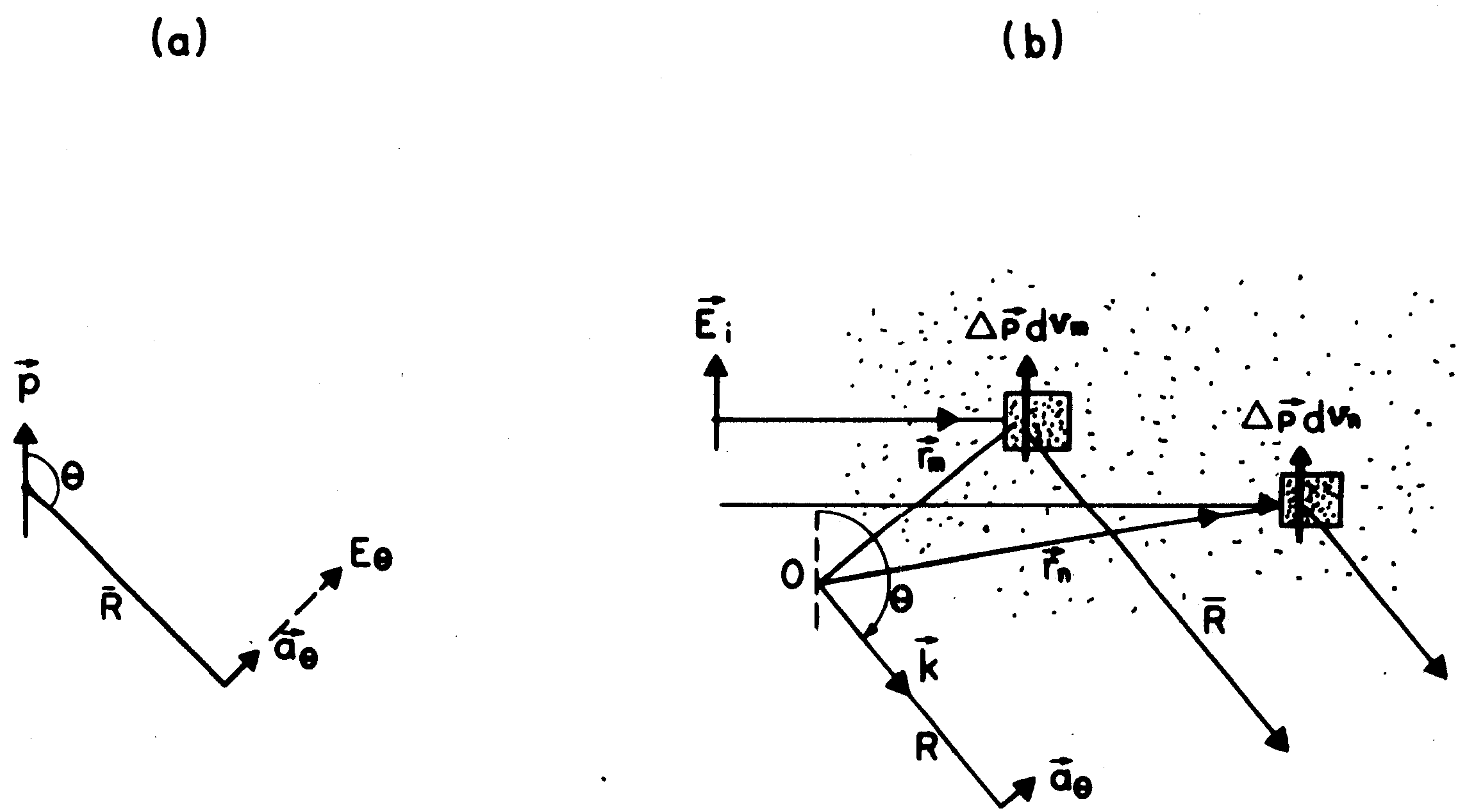

FIG. 4. (a) THE ELECTRIC FIELD OF A DIPOLE p AT LARGE DISTANCES;

(b) SUPERPOSITION OF DIPOLE FIELDS ARISING FROM PERTURBED ELEMENTS OF VOLUME.